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LETTER TO THE EDITOR

Nambu tensors and commuting vector fields

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Abstract. Takhtajan has recently studied the consistency conditions for Nambu brackets (1994 *Commun. Math. Phys.* **160** 295–315), and suggested that they have to be skew-symmetric, and satisfy the Leibnitz rule and the fundamental identity (FI, a generalization of the Jacobi identity). If the nth-order Nambu brackets in dimension N are written as $\{f_1, \ldots, f_n\} = \eta_{i_1 \ldots i_n} \partial_{i_1} f_1 \cdots \partial_{i_n} f_n$ (where the i_α summations range over $1 \ldots N$), the FI implies two conditions on the Nambu tensor η, one algebraic and one differential. The algebraic part of FI implies decomposability of η and in this letter we show that the Nambu bracket can then be written as $\{f_1, \ldots, f_n\} = \rho\, \epsilon_{\alpha_1 \ldots \alpha_n} \bar{D}^{\alpha_1} f_1 \cdots \bar{D}^{\alpha_n} f_n$, where $\epsilon_{\alpha_1 \ldots \alpha_n}$ is the usual totally antisymmetric n-dimensional tensor, the α_i summations range over $1 \ldots n$, and $\bar{D}^\alpha := \partial_\alpha + \sum_{k=n+1}^{N} v_k^\alpha \partial_k$ are n vector fields. Our main result is that the differential part of the FI is satisfied iff the vector fields $\bar{D}$ commute. Examples are provided by integrable Hamiltonian systems. It turns out that then the Nambu bracket itself guarantees that the motions stays on the manifold defined by the constants of motion of the integrable system, while the $n - 1$ Nambu Hamiltonians determine the (possibly non-integrable) motion on this manifold.

1. Introduction

The standard formulation of Hamiltonian motion using Poisson brackets is by

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{H, F\} \qquad \{f, g\} := \frac{\partial(f, g)}{\partial(p, q)}.$$

In 1973 Nambu proposed an intriguing generalization of this [1]; the idea was to extend the above classical Poisson bracket formulation in $\mathbb{R}^2$ to $\mathbb{R}^3$ by generalizing the Jacobian:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{H_1, H_2, F\} \qquad \{f, g, h\} := \frac{\partial(f, g, h)}{\partial(x, y, z)}.$$

Note the appearance of two Hamiltonians, H_1 and H_2. Subsequently, Nambu's idea has been extended further to higher dimensions (number of free variables), to higher order (number of functions in the bracket), and to other antisymmetric combinations than the Jacobian.

Recent interest in this topic is due to Takhtajan [2], who studied in particular the consistency requirements one should place on such a generalization; a natural set of properties for the bracket is as follows.

(1) Skew symmetry

$$\{f_1, \ldots, f_n\} = (-1)^{\epsilon(\sigma)} \{f_{\sigma(1)}, \ldots, f_{\sigma(n)}\}$$

where σ is a permutation of $1, \ldots, n$ and $\epsilon(\sigma)$ is its parity.

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(2) The Leibnitz rule

$$\{ab, f_2, \ldots, f_n\} = b\{a, f_2, \ldots, f_n\} + a\{b, f_2, \ldots, f_n\}.$$

(3) A generalization of the Jacobi identity, the fundamental identity (FI) (see also [3])

$$\{\{h_1, \ldots, h_{n-1}, f_1\}, f_2, \ldots, f_n\} + \{f_1, \{h_1, \ldots, h_{n-1}, f_2\}, f_3, \ldots, f_n\} + \cdots \\ \cdots + \{f_1, \ldots, f_{n-1}, \{h_1, \ldots, h_{n-1}, f_n\}\} = \{h_1, \ldots, h_{n-1}, \{f_1, \ldots, f_n\}\}. \quad (1)$$

If we write the Nambu bracket in terms of the antisymmetric Nambu tensor η [2]

$$\{f_1, \ldots, f_n\} := \eta_{i_1 \ldots i_n} \partial_{i_1} f_1 \cdots \partial_{i_n} f_n \quad (2)$$

then from the FI it follows [2] that the Nambu tensor η must satisfy two conditions, one algebraic

$$\mathcal{N}_{i_1 i_2 \ldots i_n j_1 j_2 \ldots j_n} + \mathcal{N}_{j_1 i_2 i_3 \ldots i_n i_1 j_2 j_3 \ldots j_n} = 0 \quad (3)$$

where

$$\mathcal{N}_{i_1 i_2 \ldots i_n j_1 j_2 \ldots j_n} := \eta_{i_1 i_2 \ldots i_n} \eta_{j_1 j_2 \ldots j_n} + \eta_{j_n i_1 i_3 \ldots i_n} \eta_{j_1 j_2 \ldots j_{n-1} i_2} + \eta_{j_n i_2 i_1 i_4 \ldots i_n} \eta_{j_1 j_2 \ldots j_{n-1} i_3} + \cdots \\ \cdots + \eta_{j_n i_2 i_3 \ldots i_{n-1} i_1} \eta_{j_1 j_2 \ldots j_{n-1} i_n} - \eta_{j_n i_2 i_3 \ldots i_n} \eta_{j_1 j_2 \ldots j_{n-1} i_1} \quad (4)$$

and one differential [2]

$$\mathcal{D}_{i_2 \ldots i_n j_1 \ldots j_n} := \eta_{k i_2 \ldots i_n} \, \partial_k \, \eta_{j_1 j_2 \ldots j_n} + \eta_{j_n k i_3 \ldots i_n} \, \partial_k \, \eta_{j_1 j_2 \ldots j_{n-1} i_2} + \eta_{j_n i_2 k i_4 \ldots i_n} \, \partial_k \, \eta_{j_1 j_2 \ldots j_{n-1} i_3} + \cdots \\ \cdots + \eta_{j_n i_2 i_3 \ldots i_{n-1} k} \, \partial_k \, \eta_{j_1 j_2 \ldots j_{n-1} i_n} - \eta_{j_1 j_2 \ldots j_{n-1} k} \, \partial_k \, \eta_{j_n i_2 i_3 \ldots i_n} = 0. \quad (5)$$

(In equation (5) of [2] there is a misprint in this formula (corrected in [4]): the last term of (5) above is missing.) Note that (2) and (4) are automatically satisfied for $N \leqslant n + 1$ and (5) for $N = n$.

Recently it has been shown [5] that the algebraic equations (3) and (4) imply that the Nambu tensors are decomposable (as conjectured in [4]), which in particular means that they can be written as determinants of the form

$$\eta_{i_1 \ldots i_n} = \begin{vmatrix} v^1_{i_1} & \cdots & v^1_{i_n} \\ \vdots & & \vdots \\ v^n_{i_1} & \cdots & v^n_{i_n} \end{vmatrix} = \epsilon_{\alpha_1 \ldots \alpha_n} v^{\alpha_1}_{i_1} \cdots v^{\alpha_n}_{i_1}. \quad (6)$$

In this letter, (6) is our starting point and we go on to study the consequences of differential condition (5).

2. Commuting vector fields

If η has the form (6) it actually satisfies (3) by $\mathcal{N} = 0$ and from this it follows that the differential equation (5) is scale invariant, because for any scalar ρ we have

$$\mathcal{D}_{i_2 \ldots i_n j_1 \ldots j_n}(\rho\eta) = (\rho \partial_k \rho) \, \mathcal{N}_{k i_2 \ldots i_n j_1 \ldots j_n}(\eta) + \rho^2 \, \mathcal{D}_{i_2 \ldots i_4 j_1 \ldots j_2}(\eta).$$

This scale invariance and the determinantal form of η imply certain invariances with respect to changes in the v's and we can use them to define a standard form.

Let us define an $n \times N$ matrix $\mathbf{V}$ by $\mathbf{V}_{\alpha k} := v^\alpha_k$, the Nambu tensor $\eta_{i_1 \ldots i_n}$ is then given by the determinant consisting of columns $i_1, \ldots, i_n$ of $\mathbf{V}$. The rank of $\mathbf{V}$ must be n, otherwise all η's vanish. If necessary, let us change the numbering so that the submatrix of $\mathbf{V}$ consisting of its first n columns has non-zero determinant, and let us denote this $n \times n$ submatrix by V. We have $\det V = \eta_{12 \ldots n}$.

The $n \times N$ matrix $\bar{\mathbf{V}} = V^{-1}\mathbf{V}$ can now be used to define another Nambu tensor $\bar{\eta}$ and we have simply $\eta = \det V\, \bar{\eta}$, even if the matrix entries v have changed. Since in the decomposable case the differential equations are scale invariant we may equally well consider the Nambu tensor $\bar{\eta}$. In this case we have

$$\bar{\eta}_{12\ldots n} = 1 \qquad \bar{\eta}_{\alpha_1\ldots\alpha_{n-1}k} = \epsilon_{\alpha_1\ldots\alpha_{n-1}\alpha_n}\bar{v}_k^{\alpha_n} \qquad \bar{v}_i^{\alpha} = \delta_i^{\alpha} \qquad \text{for } i \leqslant n,\ 0 \leqslant \alpha_k \leqslant n \tag{7}$$

(and, correspondingly, $\bar{v}_k^{\alpha_n} \propto \epsilon_{\alpha_1\ldots\alpha_{n-1}\alpha_n}\bar{\eta}_{\alpha_1\ldots\alpha_{n-1}k}$). This may be considered as the *standard form* for the Nambu tensor and $\bar{\mathbf{V}}$ the standard form of the defining matrix. They are quite useful in studying the differential part of FI. Furthermore, if the tensor η is given explicitly it may not be so easy to find an equally simple $\mathbf{V}$, but the entries $\bar{v}_k^{\alpha_n}$ of the standard form can be read off directly.

Using (6) we can write the Nambu bracket (2) as

$$\begin{aligned}\{f_1, \ldots, f_n\} &:= \eta_{i_1\ldots i_n}\partial_{i_1} f_1 \cdots \partial_{i_n} f_n \\ &= \epsilon_{\alpha_1\ldots\alpha_n} D^{\alpha_1} f_1 \cdots D^{\alpha_n} f_n \\ &= \rho\epsilon_{\alpha_1\ldots\alpha_n} \bar{D}^{\alpha_1} f_1 \cdots \bar{D}^{\alpha_n} f_n\end{aligned} \tag{8}$$

where

$$D^{\alpha} := \sum_{k=1}^{N} v_k^{\alpha}\partial_k \qquad \bar{D}^{\alpha} := \partial_{\alpha} + \sum_{k=n+1}^{N} \bar{v}_k^{\alpha}\partial_k \qquad \rho = \eta_{12\ldots n}. \tag{9}$$

Our main result is the following.

Theorem. The nth-order Nambu tensor in dimension N, given by (2) and (6), solves the differential condition (5) iff the differential operators $\bar{D}$ of the standard form commute.

Proof. It is clear that if the differential operators $\bar{D}$ commute they behave just like ordinary partial derivatives in computing the consequences of the FI. As noted before, the overall factor can be omitted in the decomposable case. Therefore, in this case, the Nambu tensor behaves like the canonical one and the conditions coming from FI are satisfied.

Since the Nambu tensor η changes only by an overall factor when the defining matrix $\mathbf{V}$ is multiplied by some matrix C from the left, the tensor will continue to satisfy the differential condition (5), even though in other cases the differential operators might not commute. However, from any given form of D's it is easy to go to the standard form, and what remains to be proven is that in that form the differential operators must commute. (If $n = N$ the standard form is the canonical form and there is nothing to prove.)

In the standard form $[\bar{D}^{\alpha}, \bar{D}^{\beta}] = 0$ is equivalent to

$$\partial_{\alpha}\bar{v}_l^{\beta} + \sum_{k=n+1}^{N} \bar{v}_k^{\alpha}\partial_k\bar{v}_l^{\beta} = \partial_{\beta}\bar{v}_l^{\alpha} + \sum_{k=n+1}^{N} \bar{v}_k^{\beta}\partial_k\bar{v}_l^{\alpha} \qquad \forall l > n. \tag{10}$$

Let us now take equation (5) for the case when $j_1, \ldots, j_n$ is a permutation of $1, \ldots, n$, and $i_2, \ldots, i_{n-1}$ is a permutation of an $n-2$ element subset of $1, \ldots, n$, and $i_n = l > n$ (here we need $N > n$). Since $\eta_{12\ldots n} = 1$, only the last two terms in (5) survive and we obtain the condition

$$\eta_{j_n i_2\ldots i_{n-1}k}\,\partial_k\eta_{j_1\ldots j_{n-1}l} = \eta_{j_1\ldots j_{n-1}k}\,\partial_k\eta_{j_n i_2\ldots i_{n-1}l}.$$

Contracting this with $\epsilon_{j_n i_2\ldots i_{n-1}\alpha}\epsilon_{j_1\ldots j_{n-1}\beta}$ and recalling (7) yields (10). □

In theorem 2 of [5] similar conclusions are reached but the approach of that paper is quite different.

Example. As an example let us consider $n = 3$, $N = 4$ with $\eta_{ijk} = \epsilon_{ijkl}x_l$ [4]. It is easy to see that (when $x_4 \neq 0$) in the standard form the matrix $\mathbf{V}$ is

$$\bar{\mathbf{V}} = \begin{pmatrix} 1 & 0 & 0 & -x_1/x_4 \\ 0 & 1 & 0 & -x_2/x_4 \\ 0 & 0 & 1 & -x_3/x_4 \end{pmatrix}$$

and clearly the corresponding differential operators $\bar{D}^\alpha = \partial_{x_\alpha} - (x_\alpha/x_4)\partial_{x_4}$ commute. Multiplying $\bar{\mathbf{V}}$ by x_4 produces one more alternative form $\tilde{\eta} = x_4^2\eta = x_4^3\bar{\eta}$, and the corresponding vector fields are nothing but angular momentum operators: $\tilde{D}^\alpha = L_{4\alpha} = x_4\partial_\alpha - x_\alpha\partial_4$. Now we have $[\tilde{D}^\alpha, \tilde{D}^\beta] = L_{\alpha\beta}$, but in this form the corresponding vector fields do not have to commute.

We can complete the analysis of this case by changing into new variables defined by

$$X_\alpha = x_\alpha \qquad X_4 = \tfrac{1}{2}(x_1^2 + x_2^2 + x_2^3 + x_4^2)$$

and correspondingly

$$\partial_{X_\alpha} = \bar{D}_\alpha \qquad \partial_{X_4} = \frac{1}{x_4}\partial_{x_4}.$$

All the derivative operators commute, and $\partial_{X_n} X_m = \delta_m^n$. In the new coordinates the bracket reduces to the canonical Nambu bracket almost everywhere, that is whenever $x_4 \neq 0$. In the omitted subspace one can use some other standard form. Note that derivatives with respect to the variable X_4 do not appear in the new form of the bracket, X_4 is now a constant of motion (whose form could have seen directly from the given η). Thus the motion defined by this η takes place on the sphere $x_1^2 + x_2^2 + x_2^3 + x_4^2 =$ constant, and its dynamics there is given by two Nambu Hamiltonians. The fact that the motion takes place on the surface of a hypersphere explains the appearance of angular momentum operators in the alternative form $\tilde{\eta}$. There are six such operators but only three are needed to move on the surface; the choice above was to use $L_{\alpha 4}$, which works on the chart where $x_4 > 0$ or < 0.

The above example generalizes immediately to any $N = n + 1$: if we take $\eta_{i_1\ldots i_n} = \epsilon_{i_1\ldots i_n l}(\partial_l\, m)$, the motion stays on the surface $m(x_1, \ldots, x_{n+1}) =$ constant. That this is also the most general form for $N = n+1$ (as least locally) can be seen as follows. For $N = n+1$ any η can be written as $\eta_{i_1\ldots i_n} = \epsilon_{i_1\ldots i_n l} f_l$, which means that in the standard form we have operators $\bar{D} = \partial_{x_\alpha} - f_\alpha/f_N \partial_{x_N}$ and their commutation condition is

$$\partial_\alpha g_\beta - g_\alpha\, \partial_N\, g_\beta = \partial_\beta g_\alpha - g_\beta\, \partial_N\, g_\alpha \tag{11}$$

where $g_\alpha = f_\alpha/f_N$. Now let us try to find a function m that solves $\partial_i m = k f_i$, $i = 1, \ldots, N$. From $i = N$ we get $k = \partial_N m/f_N$ so that we should solve $\partial_\alpha m = g_\alpha \partial_N m$ for $\alpha = 1, \ldots, N-1$. The integrability condition for this set of equations is nothing but (11). This means that at least locally we can find the required constant of motion m. Whether this can be done globally is another matter, and brings in the usual subtleties of chaos versus integrability.

The next generalization in this direction would be to consider $N = n + 2$ with $\eta_{i_1\ldots i_n} = \epsilon_{i_1\ldots i_n kl}(\partial_k\, f)(\partial_l\, g)$. Clearly f and g are two conserved quantities in the corresponding Nambu dynamics. In the standard form we get vector fields $\bar{D}^\alpha = \partial_\alpha - \bar{v}^\alpha_{n+2}\partial_{n+1} + \bar{v}^\alpha_{n+1}\partial_{n+2}$ with $\bar{v}^\alpha_k = (\partial_\alpha f\, \partial_k g - \partial_k f\, \partial_\alpha g)/(\partial_{n+1} f\, \partial_{n+2} g - \partial_{n+2} f\, \partial_{n+1} g)$, whose commutation can be verified directly.

3. Nambu tensors from integrable systems

With the above theorem the problem of constructing Nambu brackets has been reduced to finding commuting linear differential operators (9). A rich set of examples is now provided by integrable systems.

Let us assume that we have a Liouville integrable system in dimension n, that is we have a set of n functionally independent globally defined functions in involution, i.e. whose Poisson brackets vanish. These functions and the underlying Poisson structure define commuting $2n$-dimensional Hamiltonian vector fields ([6], section 8). With the canonical Poisson structure the Hamiltonian vector field for a function f is given by

$$F := \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \partial_{q_i} - \frac{\partial f}{\partial q_i} \partial_{p_i} \right)$$

where q_i and p_i are the canonically conjugate coordinates. Thus, if f_i are in involution for $i = 1 \ldots n$, then we can define the matrix elements of $\mathbf{V}$ as

$$v_i^j = \frac{\partial f_j}{\partial p_i} \qquad v_{i+n}^j = -\frac{\partial f_j}{\partial q_i} \qquad \forall 1 \leqslant i, j \leqslant n.$$

In this way we get nth-order Nambu tensors in dimension $N = 2n$.

3.1. Example

Let us consider the three-dimensional Toda lattice given by the Hamiltonian

$$I_2 := \tfrac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \mathrm{e}^{q_1 - q_2} + \mathrm{e}^{q_2 - q_3} + \mathrm{e}^{q_3 - q_1}.$$

This is integrable, with the other two commuting conserved quantities given by

$$I_1 := p_1 + p_2 + p_3 \tag{12}$$

$$I_3 := p_1 p_2 p_3 - \mathrm{e}^{q_2 - q_3} p_1 - \mathrm{e}^{q_3 - q_1} p_2 - \mathrm{e}^{q_1 - q_2} p_3. \tag{13}$$

The three commuting vector fields are now

$$D_T^1 = \partial_{q_1} + \partial_{q_2} + \partial_{q_3} \tag{14}$$

$$\begin{aligned} D_T^2 = {} & p_1 \partial_{q_1} + p_2 \partial_{q_2} + p_3 \partial_{q_3} + (\mathrm{e}^{q_3 - q_1} - \mathrm{e}^{q_1 - q_2}) \partial_{p_1} + (\mathrm{e}^{q_1 - q_2} - \mathrm{e}^{q_2 - q_3}) \partial_{p_2} \\ & + (\mathrm{e}^{q_2 - q_3} - \mathrm{e}^{q_3 - q_1}) \partial_{p_3} \end{aligned} \tag{15}$$

$$\begin{aligned} D_T^3 = {} & (p_2 p_3 - \mathrm{e}^{q_2 - q_3}) \partial_{q_1} + (p_1 p_3 - \mathrm{e}^{q_3 - q_1}) \partial_{q_2} + (p_1 p_2 - \mathrm{e}^{q_1 - q_2}) \partial_{q_3} \\ & + (\mathrm{e}^{q_1 - q_2} p_3 - \mathrm{e}^{q_3 - q_1} p_2) \partial_{p_1} + (\mathrm{e}^{q_2 - q_3} p_1 - \mathrm{e}^{q_1 - q_2} p_3) \partial_{p_2} \\ & + (\mathrm{e}^{q_3 - q_1} p_2 - \mathrm{e}^{q_2 - q_3} p_1) \partial_{p_3} \end{aligned} \tag{16}$$

from which the corresponding matrix $\mathbf{V}$ can be read. Since $D_T^\alpha I_\beta = 0$ for all α, β, the dynamics given by

$$\dot{g} = \{h_1, h_2, g\}_T := \epsilon_{\alpha_1 \alpha_2 \alpha_3} (D_T^{\alpha_1} h_1)\, (D_T^{\alpha_2} h_2)\, (D_T^{\alpha_3} g)$$

has the property that $\dot{I}_\alpha = 0$, no matter what the Nambu Hamiltonians h_i are.

Now recall that if an n-dimensional Hamiltonian system is Liouville integrable, then the motion actually takes place on an n-dimensional submanifold of the original $2n$-dimensional phase space defined by $I_i = c_i$, where the constants c_i are determined from the initial values. The motion on this submanifold is still defined by the original Hamiltonian.

If the dynamics is defined by a Nambu bracket arising from an integrable system as discussed above, the motion is again restricted to the manifold defined by $I_i = c_i$, but the motion on this manifold is now defined by the two additional Nambu Hamiltonians, which we could choose as we wish.

The other method of using n-dimensional integrable systems to define Nambu dynamics is to use the canonical Nambu tensor of order $2n$ and the constants of motion as Nambu Hamiltonians, for some examples see [7].

3.2. Example

If $N = 4$ integrable systems have two commuting quantities, the Hamiltonian H and a constant of motion I_2, but for a third-order Nambu tensor we would need three commuting vector fields. This is possible in some superintegrable cases, i.e. if we have one more constant of motion I_3. The third constant of motion cannot have a vanishing Poisson bracket with I_2, but the corresponding Hamiltonian vector fields could still commute. An example is provided by the following:

$$H = I_1 := F((p_1 - p_2)^2 + (q_1 - q_2)^2) \qquad I_2 := q_1 + q_2 \qquad I_3 := p_1 + p_2. \tag{17}$$

Now $\{I_2, I_3\} = 2$, but the corresponding vector fields

$$\begin{aligned}
D^1 &= F'((p_1 - p_2)^2 + (q_1 - q_2)^2)((p_1 - p_2)(\partial_{q_1} - \partial_{q_2}) - (q_1 - q_2)(\partial_{p_1} - \partial_{p_2})) \\
D^2 &= -(\partial_{p_1} + \partial_{p_2}) \\
D^3 &= \partial_{q_1} + \partial_{q_2}
\end{aligned}$$

do commute. In this case the standard form of the matrix giving the η's is quite simple, the first three columns form a unit matrix and the fourth column is given by $\bar{v}_4^1 = -\bar{v}_4^2 = (q_1 - q_2)/(p_1 - p_2)\bar{v}_4^3 = 1$. In the new variables $X_1 = q_1, X_2 = q_2, X_3 = p_1, X_4 = \frac{1}{2}(q_1 - q_2)^2 + \frac{1}{2}(p_1 - p_2)^2$, $\partial_{X_\alpha} = \bar{D}^\alpha$, for $\alpha = 1, 2, 3$, $\partial_{X_4} = 1/(p_1 - p_2)\partial_{p_2}$ the Nambu bracket reduces to the canonical one and the motion stays on the manifold $X_4 =$ constant. (In [4] the same functions were used as Hamiltonians in a fourth-order canonical Nambu bracket in dimension four.)

Another superintegrable example but with non-algebraic constants of motion is given by [8]:

$$\begin{aligned}
&H = I_1 := \tfrac{1}{2}p_1^2 + \tfrac{1}{2}p_2^2 - p_2 q_1/q_2 \qquad I_2 := (q_1 p_2 - q_2 p_1 + q_2)/p_2 \\
&I_3 := p_1 + \log(p_2/q_2).
\end{aligned} \tag{18}$$

Again $\{I_2, I_3\} = 1$, but the Hamiltonian vector fields

$$\begin{aligned}
D^1 &= p_1\partial_{q_1} + (p_2 - q_1/q_2)\partial_{q_2} + (p_2/q_2)\partial_{p_1} - (p_2 q_1/q_2^2)\partial_{p_2} \\
D^2 &= -(q_2/p_2)\partial_{q_1} + (q_2(p_1 - 1)/p_2^2)\partial_{q_2} - \partial_{p_1} + ((p_1 - 1)/p_2)\partial_{p_2} \\
D^3 &= \partial_{q_1} + (1/p_2)\partial_{q_2} + (1/q_2)\partial_{p_2}
\end{aligned}$$

commute. In the standard form the last column of $\bar{\mathbf{V}}$ is given by $\bar{v}_4^1 = p_2/(p_2 q_2 - q_1)$, $\bar{v}_4^2 = -p_2 q_1/(q_2(p_2 q_2 - q_1))$, $\bar{v}_4^3 = -p_1 q_2/(p_2 q_2 - q_1)$. The Hamiltonian defines a (non-compact) manifold on which the motion takes place, and the new variables on which the Nambu tensor is canonical are $X_1 = q_1$, $X_2 = q_2$, $X_3 = p_1$, $X_4 = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 - p_2 q_1/q_2$, with $\partial_{X_\alpha} = \bar{D}^\alpha$, for $\alpha = 1, 2, 3$, and $\partial_{X_4} = q_2/(p_2 q_2 - q_1)\partial_{p_2}$.

4. Discussion

In this letter we have studied the differential part (5) of the fundamental identity on the assumption that the algebraic part implies decomposability (6). A standard form (7)–(9) has been defined for the Nambu tensor in this case and the differential condition was related to the commutativity of the corresponding vector fields $\bar{D}$.

The simplest Nambu tensor of order n is obtained in dimension $N = n$ and is given by the totally antisymmetric constant tensor. The present results indicate that the dynamics defined by a Nambu bracket of the same order but in higher dimensions is still essentially n-dimensional.

If we define the Nambu bracket using the Hamiltonian vector field of a Liouville integrable system, then the bracket itself guarantees that the motion stays on the n-dimensional manifold defined by the constants of motion of the underlying integrable system. The motion on this manifold is determined by the Nambu Hamiltonians h_i, and this motion does not have to be integrable.

One important open problem has been the quantization of the dynamics defined by a Nambu bracket. The connection to integrable systems via commuting vector fields presented in this letter will also hopefully bring new light to this question.

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